Chapter-IV

- IIR Filters

- Design of analog filters to digital filters
  - Impulse invariance transformation
  - Bilinear Transformation
- Design of Butterworth filters
- Design of Chebyshev filters

- Digital Filters are designed by considering all the infinite samples of impulse response.
- Digital filters is a linear time-invariant discrete time systems.
- FIR filter – Non recursive type, output depends on present input & previous input samples.
- IIR filter – Recursive type output depends on the present input, past input & output samples.

- Impulse response is useful because:
  (i) any signal can be viewed as the sum of a number of shifted and scaled impulses, hence the response a linear filter to a signal is the sum of the responses to all the impulses that constitute the signal,
  (ii) an impulse input contains all frequencies with equal energy, and hence it excites a filter at all frequencies and
  (iii) impulse response and frequency response are Fourier transform pairs.

Advantages of digital filters:

- No component ageing.
- High immune to noise.
- No problems of impedance matching.
- Operated under wide range of frequencies.
- Coefficients of digital filters can be altered to obtain desired characteristics.
- Multiple filtering is possible only in digital filters.

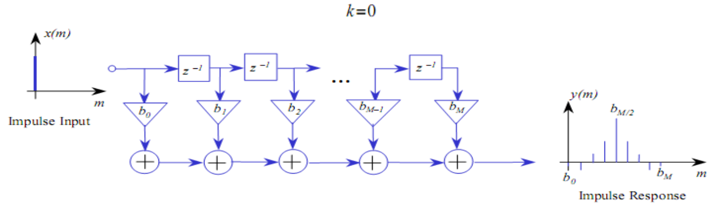
Disadvantages of digital filter:

- Quantization error arises due to finite word length in the representation of signals & parameters.

- The digital filters designed by selecting only N samples of the impulse response are called **FIR filters**.
- The digital filters designed by selecting all infinite samples of the impulse response are called **IIR filters**.

FIR filters

- A non-recursive filter has no feedback and its input-output relation is given by $y(m)=\sum_{k=0}^{M} b_k x(m-k)$



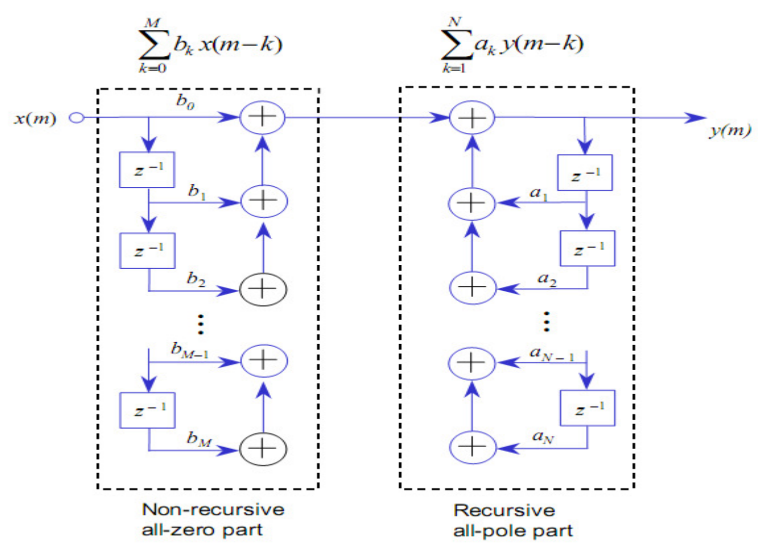
- The response of such a filter to an impulse consists of a finite sequence of M+1 samples, where M is the filter order. Hence, the filter is known as a Finite-Duration Impulse Response (FIR) filter.
- Other names for a non-recursive filter, **all-zero filter, feed-forward filter**.

IIR Filters

- A recursive filter has feedback from output to input, and in general its output is a function of the previous output samples and the present and past input samples as described by the following equation

$$y(m)=\sum_{k=1}^{N} a_k y(m-k)+\sum_{k=0}^{M} b_k x(m-k)$$

- when a recursive filter is excited by an impulse, the output persists forever. Thus a recursive filter is also known as an Infinite Duration Impulse Response (IIR) filter.
- Other names for an IIR filter - **feedback filters, pole-zero filters**.



Difference between IIR & FIR filter

IIR Filter	FIR Filter
All the infinite samples of impulse response are considered.	Only N samples of impulse response are considered.
The impulse response cannot be directly converted to digital filter transfer function.	The impulse response can be directly converted to digital filter transfer function.
The design involves analog filter design and then transforming analog filter to digital filter.	The digital filter can be directly designed to achieve the desired specifications.
The specifications include the desired characteristics for magnitude response only.	The specifications include the desired characteristics for both magnitude and phase response .
IIR filter can become unstable (if the poles of the IIR filter are outside the unit circle)	FIR filter is always stable

- Requirements of analog filter to be stable and casual:
  - Analog filter transfer function $H_a(s)$ should be a rational function of s and the coefficients of s should be real.
  - The poles should lie on the left half of s - plane.
  - Number of zeros should be less than or equal to number of poles.

- Requirements of digital filter to be stable and casual:
  - Digital filter transfer function H(z) should be a rational function of z and the coefficients of z should be real.
  - The poles should lie on the left half of z - plane.
  - Number of zeros should be less than or equal to number of poles.

- The impulse response h(n) for a realizable filter is, h(n) = 0, for n ≤ 0 & for stability it must satisfy the condition, $\sum_{n=0}^{\infty} |h(n)| < \infty$
- The design of an IIR filter for the given specifications is to find filter coefficients.

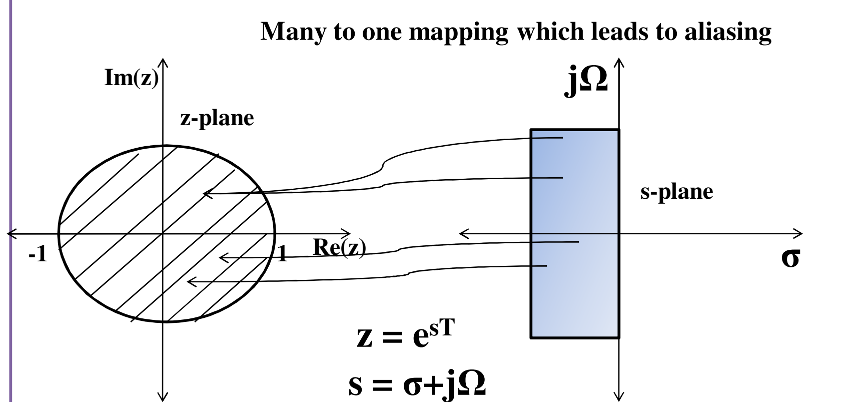
Design of IIR filters from analog filters

- Impulse invariant transformation
- Bilinear transformation

Impulse invariant transformation:

In this method the IIR filter is designed such that the unit impulse response h(n) of digital filter is the **sampled version of the impulse response of analog filter**.

Impulse invariant transformation



- There are infinite number of s-plane poles that map to the same location in the z-plane. They must have the same real parts and imaginary parts that differ by some integer multiple of 2π/T.
- This is the big disadvantage of impulse invariant mapping. The s-plane poles having imaginary parts greater, than π/T or less than - π/T causes aliasing when sampling analog signals.
- Due to the presence of aliasing, the impulse invariant method is appropriate for the design of low pass & bandpass filter only, but not suitable for HPF.

Impulse invariant transformation

Design steps:

1. For the given specifications, find $H_a(s)$ the transfer function of an analog filter.
2. Select the sampling rate of the digital filter, T seconds per sample.
3. Express the analog filter transfer function as the sum of single pole filters. $H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k}$
4. Compute the z-transform of the digital filter by using the formula,

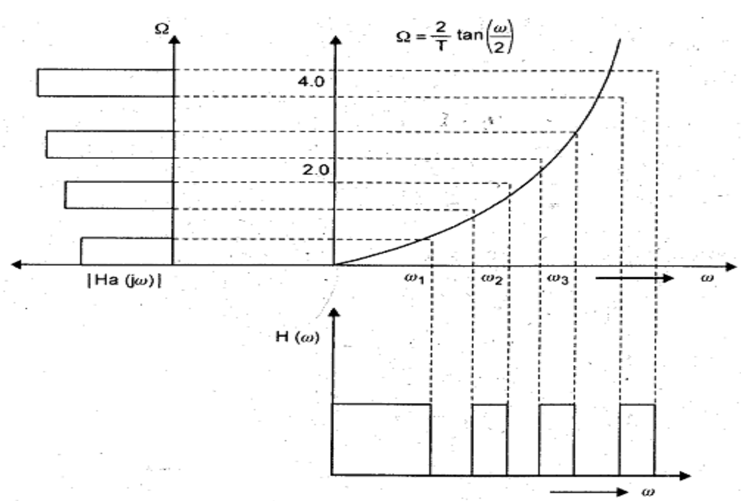
$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

Bilinear transformation

The bilinear transformation is a conformal mapping that transforms the jΩ axis into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components.

All the points in the the **LHP 's-plane' are mapped inside the unit circle of z-plane** and all the points in the RHP of 's-plane' are mapped into corresponding points outside the unit circle in the z-plane.

Frequency warping



Warping effect

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

- For small values of ω, $\Omega = \frac{2}{T}\left(\frac{\omega}{2}\right) = \frac{\omega}{T}$

$$\omega = \Omega T$$

- For low frequencies the relationship between Ω and ω are linear, as a result the digital filter have the same amplitude response as the analog filter.
- For high frequencies the relationship between Ω and ω becomes non-linear & distortion is introduced in the frequency scale of the digital filter to that of the analog filter. ⟹ Warping effect.

Prewarping

- The warping effect can be eliminated by prewarping the analog filter.
- The effect of non-linear compression at high frequencies can be compensated by prewarping.
- When the desired magnitude response is piece-wise constant over frequency, this compression can be compensated by introducing a suitable prescaling or prewarping the critical frequencies by using the formula,

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

Advantages of bilinear transformation:

1. One – to - one mapping.
2. Stable continuous systems can be mapped into realizable, stable digital systems.
3. No aliasing.

Disadvantages:

1. Highly non-linear, producing frequency compression at high frequencies.

Bilinear transformation

Design steps:

1. Find prewarping analog frequencies using

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

2. Using analog frequency, find H(s) of the analog filter.
3. Select the sampling rate of the digital filter, T = 1sec per sample.
4. Substitute, $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$ into the transfer function found in step 2.

Ex. 5.12(RB): For the analog transfer function

$$H(s)=\frac{2}{(s+1)(s+2)}$$

Determine H(z) using impulse invariance method. Assume T = 1 sec.

Solution:

$$H(s)=\frac{2}{(s+1)(s+2)}$$

Applying partial fractions,

$$H(s)=\frac{2}{(s+1)}-\frac{2}{(s+2)}$$

$$=\frac{2}{(s-(-1))}-\frac{2}{(s-(-2))}$$

$$H_a(s)=\sum_{k=1}^{N}\frac{c_k}{s-p_k} \Rightarrow H(z)=\sum_{k=1}^{N}\frac{c_k}{1-e^{p_k T}z^{-1}}$$

$$H(s)=\frac{2}{(s-(-1))}-\frac{2}{(s-(-2))}$$

$$(s-p_k)\rightarrow\left(1-e^{p_k T}z^{-1}\right)$$

Here , $p_k \rightarrow p_1, p_2$

$p_1=-1;\quad p_2=-2;\quad T=1\text{sec}$

$$H(z) = \frac{2}{1-e^{-1}z^{-1}} - \frac{2}{1-e^{-2}z^{-1}}$$
$$= \frac{2}{1-0.3678z^{-1}} - \frac{2}{1-0.1353z^{-1}}$$

Simplifying,

$$H(z) = \frac{0.465z^{-1}}{1-0.503z^{-1}+0.0497z^{-2}}$$

Ex. 5.11(RB): Apply bilinear transformation to

$$H(s) = \frac{2}{(s+1)(s+2)}$$

with T = 1sec and find H(z).

Solution:

Substitute,

$$s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$H(z) = \left.\frac{2}{(s+1)(s+2)}\right|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$H(z)=\frac{2}{\left(2\left[\frac{1-z^{-1}}{1+z^{-1}}\right]+1\right)\left(2\left[\frac{1-z^{-1}}{1+z^{-1}}\right]+2\right)}$$

$$=\frac{\left(1+z^{-1}\right)^{2}}{6\left(1-\frac{1}{3}z^{-1}\right)}$$

Design procedure of Low pass Butterworth IIR filter

Designing IIR digital filter involves the design of equivalent analog filter & then converting analog filter to digital filter.

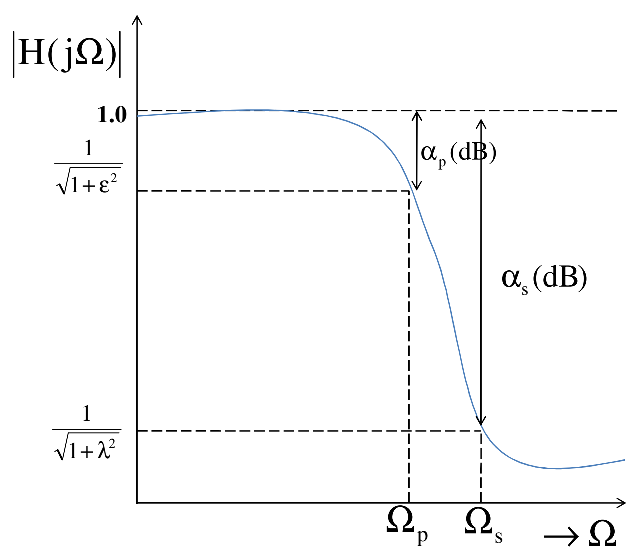
- First analog Butterworth IIR filter transfer function is determined using the specifications.
- Then, analog transfer function is converted to a digital filter transfer function using,
  - Impulse invariance transformation
  - Bilinear transformation

Analog Butterworth filter:

- The magnitude response of LPF is given by,

$$|H_a(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

- The magnitude response of the Butterworth filter is said to be maximally flat.



Design procedure of Low pass Butterworth IIR filter

- Find order of the filter,

$$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \geq \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\left(\frac{1}{k}\right)}$$

ε – parameter specifying allowable passband
λ - parameter specifying allowable stopband

$$\frac{1}{\sqrt{1+\varepsilon^2}} = \alpha_p \qquad \frac{1}{\sqrt{1+\lambda^2}} = \alpha_s$$

- Obtain cut-off frequency,

$$\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p} - 1\right)^{1/2N}} = \frac{\Omega_p}{(\varepsilon)^{1/2N}}$$

- With the order of the filter, obtain the transfer function H(s), by substituting s by $\frac{s}{\Omega_c}$
- Convert analog transfer function $H_a(s)$ to digital function H(z), using either impulse invariance method or bilinear transformation method.

How to obtain the poles(denominator) of the transfer function?

To obtain the poles which lies on LHS of the s-plane,

$$s_k = e^{j\phi k}; \quad \text{where} \quad \phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$$

When N = 1; k = 1

$$s_1 = e^{j\phi_1}; \quad \text{where} \quad \phi_1 = \frac{\pi}{2} + \frac{(2-1)\pi}{2} = \pi$$

$$s_1 = e^{j\pi} = -1$$

Denominator: → [s - (-1)] = (s+1)

How to obtain the poles(denominator) of the transfer function?

$$s_k = e^{j\phi_k}; \quad \text{where} \quad \phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$$

When N = 2; k = 1, 2

$$s_1 = e^{j\phi_1}; \quad \text{where} \quad \phi_1 = \frac{\pi}{2} + \frac{(2-1)\pi}{4} = \frac{3\pi}{4}$$

$$s_1 = -0.707 + j0.707$$

$$s_2 = e^{j\phi_2}; \quad \text{where} \quad \phi_2 = \frac{\pi}{2} + \frac{(4-1)\pi}{4} = \frac{5\pi}{4}$$

$$s_2 = -0.707 - j0.707$$

Den. → [s - (-0.707+j0.707)] [s - (-0.707-j0.707)]
= $(s^2+1.414s+1)$

List of Butterworth polynomials:

N	Denominator of H(s)
1	(s+1)
2	$s^2+\sqrt{2}s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.76537s+1)(s^2+1.8477s+1)$

- For the given specifications design an analog Butterworth filter,

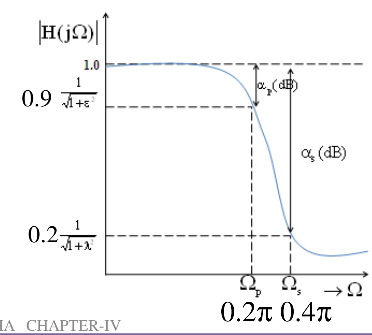
$$0.9 \le |H(j\Omega)| \le 1, \quad \text{for} \quad 0 \le \Omega \le 0.2\pi$$
$$|H(j\Omega)| \le 0.2, \quad \text{for} \quad 0.4\pi \le \Omega \le \pi$$

Solution:

$$\Omega_p = 0.2\pi; \quad \Omega_s = 0.4\pi$$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.9 \Rightarrow \varepsilon = 0.4843$$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2 \Rightarrow \lambda = 4.899$$



$$N \geq \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \geq 3.34$$

$$N = 4$$

$$H(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)}$$

- Find the cut-off frequency:

$$\Omega_c = \frac{\Omega_p}{\varepsilon^{1/N}} = 0.24\pi$$

- obtain the transfer function H(s), by substituting s by $\frac{s}{\Omega_c} = \frac{s}{0.24\pi}$

$$H(s) = \frac{1}{\left\{\left(\frac{s}{0.24\pi}\right)^2 + 0.76537\left(\frac{s}{0.24\pi}\right) + 1\right\}\left\{\left(\frac{s}{0.24\pi}\right)^2 + 1.8477\left(\frac{s}{0.24\pi}\right) + 1\right\}}$$

$$H(s) = \frac{0.323}{(s^2 + 0.577s + 0.0576\pi^2)(s^2 + 1.393s + 0.0576\pi^2)}$$

- Design an analog Butterworth filter that has a -2dB passband attenuation at a frequency of 20rad/sec an atleast -10dB stopband attenuation at 30rad/sec.

Solution:

$$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \geq 3.37 \Rightarrow N = 4$$

$$H(s) = \frac{1}{(s^2+0.76537s+1)(s^2+1.8477s+1)}$$

$$\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p}-1\right)^{1/2N}} = 21.3868$$

- obtain the transfer function H(s), by substituting s by $\frac{s}{\Omega_c} = \frac{s}{21.3868}$

$$H(s) = \frac{0.20921\times 10^6}{(s^2+16.3686s+457.394)(s^2+39.5176s+457.394)}$$

- **Ex. 5.32RB:** Design Butterworth filter using the impulse invariance method for the following specifications: $0.8 \le |H(e^{j\omega})| \le 1, \quad 0 \le \omega \le 0.2\pi$

$$|H(e^{j\omega})| \le 0.2, \; 0.6\pi \le \omega \le \pi$$

Solution:

Given,

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.8 \Rightarrow \varepsilon = 0.75$$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2 \Rightarrow \lambda = 4.899$$

$\omega_s = 0.6\pi rad; \quad \omega_p = 0.2\pi rad; \quad$ Assume $T = 1 sec.$

$$\frac{\omega_s}{\omega_p} = \frac{\Omega_s T}{\Omega_p T} = \frac{\Omega_s}{\Omega_p} = \frac{0.6\pi}{0.2\pi} = 3$$

$$N \ge \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \ge \frac{\log\left(\frac{4.899}{0.75}\right)}{\log 3} \ge 1.71$$

$$N = 2$$

- For N = 2, the transfer function of normalized Butterworth filter is,

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

- Obtain cut-off frequency, $\Omega_c = \frac{\Omega_p}{(\varepsilon)^{1/N}} = \frac{0.2\pi}{(0.75)^{1/N}} = 0.231\pi$

$$H_a(s) = H(s)\Big|_{s \to \frac{s}{\Omega_c}}$$

$$= \frac{0.5266}{s^2 + 1.03s + 0.5266}$$

$$= \frac{0.5266}{(s + 0.51 + j0.51)(s + 0.51 - j0.51)}$$

- Applying partial fractions,

$$H_a(s) = \frac{j0.516}{s - (-0.51 - j0.51)} - \frac{j0.516}{s - (-0.51 + j0.51)}$$

- Using impulse invariance method,

$$\text{If } H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k}, \quad \text{then } H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

$$H(z) = \frac{j0.516}{1 - e^{-0.51 - j0.51} z^{-1}} - \frac{j0.516}{1 - e^{-0.51 + j0.51} z^{-1}}$$

$$= \frac{j0.516}{1 - e^{-0.51} e^{-j0.51} z^{-1}} - \frac{j0.516}{1 - e^{-0.51} e^{j0.51} z^{-1}}$$

$$= \frac{j0.516(1 - e^{-0.51} e^{j0.51} z^{-1}) - j0.516(1 - e^{-0.51} e^{-j0.51} z^{-1})}{(1 - 0.6e^{-j0.51} z^{-1})(1 - 0.6e^{j0.51} z^{-1})}$$

$$= \frac{j0.516 - j0.516e^{-0.51} e^{j0.51} z^{-1} - j0.516 + j0.516e^{-0.51} e^{-j0.51} z^{-1}}{1 + 0.36e^{-j0.51} e^{j0.51} z^{-2} - 0.6e^{-j0.51} z^{-1} - 0.6e^{j0.51} z^{-1}}$$

$$= \frac{j0.3096[0.873 + j0.488]z^{-1} - j0.3096[0.873 - j0.488]z^{-1}}{1 - 0.6[e^{-j0.51} + e^{j0.51}]z^{-1} + 0.36e^{-j0.51}e^{j0.51}z^{-2}}$$

$$\therefore\ H(z) = \frac{0.3022}{1\text{-}1.047z^{-1} + 0.36z^{-2}}$$

- **Ex. 5.33:** Design a Butterworth filter using the bilinear transformation for the above problem.

Solution:

$$\omega_s = 0.6\pi \text{rad};\quad \omega_p = 0.2\pi \text{rad. Assume } T = 1\sec.$$

Prewarping the frequencies we get,

$$\Omega_s = 2\tan\frac{\omega_s}{2} = 2.752;\quad \Omega_p = 2\tan\frac{\omega_p}{2} = 0.6498$$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.8 \Rightarrow \varepsilon = 0.75;\quad \frac{1}{\sqrt{1+\lambda^2}} = 0.2 \Rightarrow \lambda = 4.899$$

$$\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 4.235$$

$$N \geq \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\left(\frac{1}{k}\right)} \geq \frac{\log\left(\frac{4.899}{0.75}\right)}{\log(4.235)} \geq 1.3$$

$$N = 2$$

- For N = 2, the transfer function of normalized Butterworth filter is,

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

- Obtain cut-off frequency, $\Omega_c = \frac{\Omega_p}{(\varepsilon)^{1/N}} = \frac{0.2\pi}{(0.75)^{1/2}} = 0.75\text{rad/sec}$

$$H_a(s) = H(s)\Big|_{s \to \frac{s}{\Omega_c}}$$

$$= \frac{0.5625}{s^2 + 1.06s + 0.5625}$$

- For a bilinear transformation,

$$H(z) = H_a(s)\Big|_{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)};$$

Assume T = 1sec.

$$H(z)=\frac{0.5625(1+z^{-1})^2}{4(1-z^{-1})^2+2.12(1-z^{-2})+0.5625(1+z^{-1})^2}$$

$$=\frac{0.084(1+z^{-1})^2}{1-1.028z^{-1}+0.3651z^{-2}}$$

Design of Chebyshev filter:

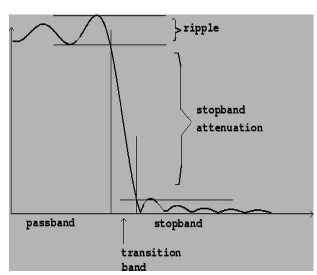
- Order of the filter:

$$N \geq \frac{\cosh^{-1}\left(\frac{\lambda}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

- Values of a and b – minor axis and major axis respectively,

$$a=\Omega_p\frac{\left[\mu^{1/N}-\mu^{-1/N}\right]}{2};\ b=\Omega_p\frac{\left[\mu^{1/N}+\mu^{-1/N}\right]}{2}$$

$$\text{where, } \mu=\varepsilon^{-1}+\sqrt{\varepsilon^{-2}+1};\ \varepsilon=\sqrt{10^{0.1\alpha_p}-1}$$



- Calculate the poles of Chebyshev filter which lie on an ellipse by using the formula,

$$S_k = a\cos\phi_k + jb\sin\phi_k;\quad k = 1,2,...N$$

$$\text{where, } \phi_k = \frac{\pi}{2} + \left[\frac{2k-1}{2N}\right]\pi;\ k = 1,2,...N$$

- Find the denominator polynomial of the transfer function using the above poles.

- The **numerator value** of the transfer function depends on the value of N.
  - For **odd N, sub. s = 0** in the denominator polynomial & find the value which is the numerator of the transfer function.
  - For **even N, sub. s = 0** in the denominator polynomial & **divide the value by** $\sqrt{1+\varepsilon^2}$ which is the numerator of the transfer function.

- **Ex. 5.34:** Design a Chebyshev filter for the following specification using (a) Bilinear transformation (b) impulse invariance method. $0.8 \le |H(e^{j\omega})| \le 1, \quad 0 \le \omega \le 0.2\pi$

$$|H(e^{j\omega})| \le 0.2, \; 0.6\pi \le \omega \le \pi$$

Solution: (a). Using bilinear transformation

$$\omega_s = 0.6\pi rad; \;\; \omega_p = 0.2\pi rad. \;\; \text{Assume } T = 1 sec.$$

Prewarping the frequencies,

$$\Omega_s = 2\tan\frac{\omega_s}{2} = 2.752; \;\; \Omega_p = 2\tan\frac{\omega_p}{2} = 0.6498$$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.8 \Rightarrow \varepsilon = 0.75; \; \frac{1}{\sqrt{1+\lambda^2}} = 0.2 \Rightarrow \lambda = 4.899$$

$$\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 4.235$$

$$N \ge \frac{\cosh^{-1}\left(\frac{\lambda}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{2.564}{2.122} = 1.208$$

$$N = 2$$

- **To find a & b:**

$$\mu = \varepsilon^{-1} + \sqrt{\varepsilon^{-2} + 1} = 3$$

$$a = \Omega_p \frac{\left[\mu^{1/N} - \mu^{-1/N}\right]}{2} = 0.3752$$

$$b = \Omega_p \frac{\left[\mu^{1/N} + \mu^{-1/N}\right]}{2} = 0.75$$

- **To find the poles:**

$$s_k = a\cos\phi_k + jb\sin\phi_k; \quad k = 1,2,\ldots N$$

$$\text{where, } \phi_k = \frac{\pi}{2} + \left[\frac{2k-1}{2N}\right]\pi; \quad k = 1,2,\ldots N$$

$$\phi_1 = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4} = 135;$$

$$\phi_2 = \frac{\pi}{2} + \frac{3\pi}{4} = \frac{5\pi}{4} = 225.$$

$$s_1 = -0.2653 + j0.53 \quad \& \quad s_2 = -0.2653 - j0.53$$

- **Denominator of H(s):**

$$(s + 0.2653 - j0.53)(s + 0.2653 + j0.53)$$

$$= (s + 0.2653)^2 + (0.53)^2$$

10/5/2012 S. THAI SUBHA CHAPTER-IV 57

- For even N, sub. s = 0 in the denominator polynomial & divide the value by $\sqrt{1+\varepsilon^2}$ which is the numerator of the transfer function.

$$\text{Den. polynomial : } \frac{(s+0.2653)^2 + (0.53)^2\Big|_{s=0}}{\sqrt{1+\varepsilon^2}}$$

$$\text{Numerator of } H(s) = \frac{0.3516}{\left[1+(0.75)^2\right]^{1/2}} = 0.28$$

$$H(s) = \frac{0.28}{s^2 + 0.5306s + 0.3516}$$

Using bilinear transformation

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \qquad [\because T = 1 \text{ sec.}]$$

$$H(z) = \frac{0.28(1+z^{-1})^2}{5.4128 - 7.298z^{-1} + 3.29z^{-2}} = \frac{0.052(1+z^{-1})^2}{1 - 1.3480z^{-1} + 0.608z^{-2}}$$

10/5/2012 S. THAI SUBHA CHAPTER-IV 58

(b) Impulse Invariance Method:

$$\omega = \Omega T$$

$$\omega_p = \Omega_p T \text{ and } \quad \omega_s = \Omega_s T$$

For T = 1 sec.

$$\frac{\omega_s}{\omega_p} = \frac{\Omega_s}{\Omega_p} = \frac{0.6\pi}{0.2\pi} = 3$$

$$N = \frac{\cosh^{-1}(\lambda/\varepsilon)}{\cosh^{-1}(1/k)} = \frac{\cosh^{-1}\frac{4.899}{0.75}}{\cosh^{-1}(3)} = 1.45 \approx 2$$

$$\mu = \varepsilon^{-1} + \sqrt{\varepsilon^{-2} + 1} = 3$$

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 0.3627$$

$$b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 0.7255$$

$\phi_1 = 135°, \phi_2 = 225$

$s_1 = -0.2564 + j\,0513$

$s_2 = -0.2564 - j\,0.513$

Denominator of H (s) $= s^2 + 0.513s + 0.33$

Numerator of H (s) $= 0.264$

$$H(s) = \frac{0.264}{s^2 + 0.513s + 0.33}$$

$$H(s) = \frac{0.263}{(s + 0.257 + j0.515)(s + 0.257 - j0.515)}$$

- Applying partial fractions,

$$H(s) = \frac{j0.257}{(s + 0.257 + j0.515)} - \frac{j0.257}{(s + 0.257 - j0.515)}$$

$$\text{If } H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k}, \quad \text{then } H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

$$P_1 = -0.257 - j0.515$$
$$P_2 = -0.257 + j0.515$$

$$H(z) = \frac{j0.257}{1 - e^{-0.257} e^{-j0.515} z^{-1}} - \frac{j0.257}{1 - e^{-0.257} e^{j0.515} z^{-1}}$$

$$= \frac{j0.257\left[1 - e^{-0.257} e^{j0.515} z^{-1}\right] - j0.257\left[1 - e^{-0.257} e^{-j0.515} z^{-1}\right]}{\left[1 - 0.77e^{-j0.52} z^{-1}\right]\left[1 - 0.77e^{j0.52} z^{-1}\right]}$$

$$= \frac{-j0.198e^{j0.515} z^{-1} + j0.198e^{-j0.515} z^{-1}}{1 + 0.5929z^{-2} - 1.34z^{-1}}$$

$$= \frac{-j0.198z^{-1}\left[e^{j0.515} - e^{-j0.515}\right]}{1 + 0.5929z^{-2} - 1.34z^{-1}} = \frac{-j0.198z^{-1}\left[2j\sin 0.52\right]}{1 + 0.5929z^{-2} - 1.34z^{-1}}$$

$$H(z) = \frac{0.1967z^{-1}}{1 + 0.5929z^{-2} - 1.34z^{-1}}$$